A REDUCED ORDER MODEL OF WAVE PROPAGATION AND VIBRATIONS IN RODS AND TUBES OF UNIFORM CROSS SECTIONS

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RESUMEN

After a short historical survey of some work done on the linear theory of longitudinal vibrations and wave propagation in rods and tubes of uniform cross-section, a reduced order mathematical model (ROM) for wave propagation and vibrations in rods and tubes is proposed selecting three propagation modes (one extensional and two shear modes) with dispersion relations corresponding to mixed boundary conditions. These modes are coupled in order to approximately comply with zero-stress boundary conditions. The coupling gives a set of operator equations in the mode amplitudes, with time and a single space coordinate (along the axis of symmetry of the rod or tube) as independent variables: these are the ROM's equations. The model is general enough to describe vibrations and wave propagation in rods and tubes made of linear hereditary solids. The analysis of wave propagation and vibrations is then restricted to linear elastic materials, focused in either very low frequency or very high frequency phenomena. Analytical formulae for group velocities are derived, as well as asymptotic expressions for the propagation of mode amplitudes. A simplified theoretical framework to study mechanical resonance in bars of any cross section is derived. A qualitative comparison with experimental results is done. The limitations and pitfalls of the model are assessed. New experiments and digital simulations are suggested to test some of its predictions.

Palabras Clave: mechanical vibrations, wave propagation, hereditary-elastic materials, propagation modes in rods and tubes, reduced order models.

1. INTRODUCTION

(1.1)-Longitudinal vibrations and wave propagation in rods and tubes, by themselves or coupled with torsional or flexural wave motion, appear in the classical fields of civil, mechanical, electrical and aerospace engineering, of non-destructive testing, and in the new fields of mechatronics and micro-electromechanical systems.

The mathematical study of longitudinal vibrations began almost three centuries ago.
For an infinite cylindrical and linear elastic waveguide, with zero stress boundary conditions and assuming a purely sinusoidal time dependence, the abovementioned equations of the dynamic theory of elasticity were posed in cylindrical coordinates and solved by separation of variables by Pochhammer in 1876 (Kolsky, 1963, pp.54-59). The boundary conditions then give a transcendental equation that relates the **angular frequency** ω with the **wave number** k along the axis of the rod: the so called dispersion relation, in terms of Bessel functions. Assuming a suitable spatial symmetry (to eliminate flexural and torsional vibrations) a numerable infinity of propagation modes is obtained from the aforementioned dispersion relation.

Each mode is characterized by a definite connection between ω and k, as well as by certain typical space patterns of strain and stress. In principle, by superposition of the contributions of the different modes and the different frequencies that belong to an elastic pulse or wave pattern, we can follow the propagation and distortion of the pattern along the axis of a bar.
In the case of finite cylinders, realistic boundary conditions at the end (plane) cross sections, introduced seemingly insurmountable additional complications to obtain a solution by separation of variables in the partial differential equations (Nadeau, 1964, pp.268-271).
This analytical approach could be extended to cylindrical tubes, but always with increasing difficulties. However, some work along these lines was done at the end of the fifties and during the sixties of the twentieth century, including some extensions to wave propagation and vibrations of hereditary materials in rods.
It was soon realized that it may be easier to make an ab-initio numerical analysis without the introduction of the mode concept and without spectral decomposition of the wave pattern.
This is the contemporary trend, fostered by the increase in computer power and the development of high-quality numerical algorithms.
But even with the present computing facilities, the problem of longitudinal vibrations and wave propagation can be difficult to tackle in certain cases of practical interest, even if internal friction is neglected.
One disadvantage of digital simulations is that, sometimes, they don't show clearly the effects of changing the physical parameters in the problem.
So, a simplified analytical approach may be of interest, both as a guide for the design of ab-initio digital simulations of wave propagation or vibrations, and for the interpretation of experimental results.
The main purpose of this work is to propose a general reduced order model (ROM) for longitudinal vibration and wave propagation in bars or tubes that can be applied both to linear elastic and hereditary materials.
Then linear elastic materials are considered.
The results already obtained by the author in relation with high frequency wave propagation and low frequency vibrations in rods are briefly reviewed.

(1.2)-The key concept for ROM construction in acoustic of solids (elastic waves and vibrations) is a phenomenon very important in many branches of engineering and physics: **wave coupling** (Pierce, 1954).
Let us summarize its main features. If the dispersion curves of two uncoupled modes of propagation cross each other one or several times, when a phenomenological coupling between them is introduced, the resulting dispersion curves for the coupled modes split from the crossover points. If the phenomenological coupling is mild, the new dispersion curves nearly coincide, far from these intersection points, with part of the curves corresponding to the uncoupled modes. However, if the coupling is strong enough, a coupled mode may behave very differently from an uncoupled one, even far from the abovementioned cross section points.
All these properties of wave coupling will be used to construct the reduced order mathematical model proposed in this paper.
But let us make first a brief review of the previous work related with the construction of simple analytical models of longitudinal vibrations and waves in rods.

Modal analysis of wave propagation is amenable to an analytical approach, as the one intended here, if it can be done using only a few modes (two or three).
There is another idea that we will borrow in order to construct a simplified mathematical model for longitudinal wave propagation. According to the results obtained by the computing group of Bell Telephone Laboratories, in connection with early delay line studies (Mc Skimin, 1956), the dispersion relation of the lowest mode of a fluid waveguide behaves very similar to the dispersion relation of the lowest mode corresponding to predominantly longitudinal waves, in the solid rod, as predicted by the equations of dynamics elasticity.
This suggested the introduction of the concept of "**almost-fluid waveguide**" to describe with a maximum of simplicity the propagation of high frequency longitudinal waves in solid bars (Suárez-Antola, 1990 and 1998, Suárez-Ántola and Suárez-Bagnasco, 1999).
Then, in order to comply with the almost-fluid waveguide concept, we will require that the dispersion relation for the highest mode of propagation of our intended model must be: (a) asymptotic to $\omega = v_L.k$ when k tends to infinity, and (b) near the dispersion relation of the first mode of the almost fluid waveguide for k big enough.
Besides a **lower extensional-surface mode** and **higher shear-dilatational modes**, the exact dispersion equation shows **a third kind of propagation mode** (Mc Skimin, 1956; Kolsky, 1963). The phase velocity is in this case asymptotic to v_T when k tends to infinity (asymptotic shear mode). But the group velocity of the main asymptotic shear mode, considered as a function of $k.R$ (being R the bar radius), presents a significant maximum (less than v_E) followed by a less significant minimum (higher than the minimum of the group velocity of the extensional surface mode) before approaching asymptotically to v_T (Kolsky, 1963, Fig.15).
As a consequence of all this, to study the propagation of short longitudinal pulses (like very short ultrasonic pulses with a wide spectrum of frequencies), we will require three branches in the dispersion equation obtained from our model: **a lower extensional-surface branch, an asymptotically shear branch in the middle, and an upper asymptotically dilatational branch** (Suárez-Ántola and Suárez-Bagnasco, 2001).

2. METHODS OF ROM CONSTRUCTION

Let us begin by considering a pure extensional mode with dispersion equation $\omega = v_E.k$, and two shear modes with dispersion equations:

$$\omega^2 = \omega_{c1}{}^2 + v_T{}^2.k^2 \qquad\qquad \omega^2 = \omega_{c2}{}^2 + v_T{}^2.k^2$$

For a rod the cut-off frequencies are given by the equations:

$$\omega_{c1} = \frac{\alpha_{T1} \cdot v_T}{R} \qquad\qquad \omega_{c2} = \frac{\alpha_{T2} \cdot v_T}{R}$$

α_{T1} and α_{T2} are pure numbers, with α_{T1} less than α_{T2}. For the two lower and usually dominant modes we have $\alpha_{T1} = 3.83$ and $\alpha_{T2} = 7.02$.
These dispersion equations for the shear modes are obtained after solving the equations of elasto-dynamics for an infinite waveguide of circular cross section, using mixed boundary conditions instead of the stress-free boundary conditions imposed by Pochhammer and Chree (Auld, 1973).

Mixed boundary conditions means here that the radial component of displacement s_r and the axial component of stress $\sigma_{r,z}$ are both zero at the boundary of the waveguide, that is, when r = R.
The extensional mode also verifies mixed boundary conditions, because in this case only s_z and $\sigma_{z,z}$ may be different from zero.
However, this mode is not an exact solution of the full equations of dynamics elasticity, since it neglects the lateral contraction or dilatation due to Poisson´s effect.
As v_E is greater than v_T, we see that the curves that represent the dispersion equations of the shear modes intersect the straight line that represents the dispersion equation of the extensional mode in the ω-k plane.
Now, the **strong coupling** of intersecting modes obtained using mixed boundary conditions produce new emerging modes that can be used to describe several features of the case with stress-free boundary conditions (Achenbach, 1973; Auld, 1973).
However, each shear mode intersects the extensional mode (the extensional mode couples with both shear modes), but the shear modes don't intersect (each shear mode couples only with the extensional mode).
So, let us construct a model for the propagation of longitudinal pulses in linear elastic rods of uniform transverse sections, coupling three one dimensional equations that correspond to intersecting modes of propagation, as follows:

$$\rho \frac{\partial^2 A}{\partial t^2} = \frac{\partial \sigma}{\partial z} + \rho . f \qquad (1a)$$

$$\rho \frac{\partial^2 B_1}{\partial t^2} + \rho {\omega_{c,1}}^2 B_1 = \frac{\partial \tau_1}{\partial z} + \rho . g_1 \qquad (1b)$$

$$\rho \frac{\partial^2 B_2}{\partial t^2} + \rho {\omega_{c,2}}^2 B_2 = \frac{\partial \tau_2}{\partial z} + \rho . g_2 \qquad (1c)$$

The fields $A(t,z)$, $B_1(t,z)$ and $B_2(t,z)$ are certain coupled mode amplitudes that describe mechanical wave propagation and vibrations.
In Equations (1), the stresses σ and τ are **equivalent** normal stress and **equivalent** shear stresses.
The additional terms $\rho . f(t,z)$ $\rho . g_1(t,z)$ $\rho . g_2(t,z)$ represent distributed forces that may excite the corresponding propagation mode. For example: magneto-strictive, electro-strictive or thermo-elastic forces, amongst others.
The effect of forces due to transducers or other excitation devices located at the ends of the rods or tubes appear in the boundary conditions, as usual.
The terms $\rho . {\omega_c}^2 . B$ look like a kind of elastic restoring forces associated with the displacements B, and they are added to the spatial derivatives of the "equivalent stresses" σ , τ and to the volume forces $\rho . f$ and $\rho . g$ in the above coupled set of non-homogeneous partial differential equations.
Equations (1) are already in the form often used in the study of mode coupling in engineering.
In linear rods of uniform transverse sections, the **equivalent** normal stress σ and the **equivalent** shear stresses τ are given by the relations:

$$\sigma = \hat{E}.\left(\frac{\partial A}{\partial z} - K_1.\frac{\partial B_1}{\partial z} - K_2 \cdot \frac{\partial B_2}{\partial z} \right) \qquad (2a)$$

$$\tau_1 = \hat{G}.\left(\frac{\partial B_1}{\partial z} - K_1.\frac{\partial A}{\partial z} \right) \qquad (2b)$$

$$\tau_2 = \hat{G}.\left(\frac{\partial B_2}{\partial z} - K_2.\frac{\partial A}{\partial z} \right) \qquad (2c)$$

The symbols $\hat{E}$ and $\hat{G}$ represent certain operators that will be detailed below.
Here K_1 , K_2 are phenomenological coupling constants. They will be suitably restricted later. If $K_1 = K_2 = 0$, the coupling disappears and $A(t,z)$ becomes the amplitude of the pure extensional mode, while $B_1(t,z)$ and $B_2(t,z)$ becomes the amplitudes of suitable pure shear modes. In general, when both $K_\alpha \neq 0$ we obtain three equations that couple the the **equivalent** normal stress σ and the **equivalent** shear stresses τ with the amplitudes $A(t,z)$, $B_\alpha(t,z)$
The linear elastic case is obtained from Equations (1) and (2) introducing the Young modulus $\hat{E} = E_0 \cdot \hat{I}$ and shear modulus $\hat{G} = G_0 \cdot \hat{I}$ (being $\hat{I}$ the identity operator). Then, if the relations $v_E = \sqrt{E_0/\rho}$ for the extensional velocity and $v_T = \sqrt{G_0/\rho}$ for the shear wave velocity in a linear elastic solid are taken into account, it is possible to derive from Equations (1) and (2) the following coupled set of wave equations:

$$\frac{\partial^2 A}{\partial z^2} - \frac{1}{v_E^2}\frac{\partial^2 A}{\partial t^2} = K_1 \frac{\partial^2 B_1}{\partial z^2} + K_2 \frac{\partial^2 B_2}{\partial z^2} \qquad (3a)$$

$$\frac{\partial^2 B_1}{\partial z^2} - \frac{1}{v_T^2}\frac{\partial^2 B_1}{\partial t^2} - \frac{\omega_{c1}^2}{v_T^2} B_1 = K_1 \frac{\partial^2 A}{\partial z^2} \qquad (3b)$$

$$\frac{\partial^2 B_2}{\partial z^2} - \frac{1}{v_T^2}\frac{\partial^2 B_2}{\partial t^2} - \frac{\omega_{c2}^2}{v_T^2} B_2 = K_2 \frac{\partial^2 A}{\partial z^2} \qquad (3c)$$

Let us search new emerging modes of propagation substituting the following **ansatz** in Equations (3):

$$A(t,z) = C(k) \cdot e^{j(\omega \cdot t - k \cdot z)} \qquad (4a)$$

$$B_1(t,z) = D_1(k) \cdot e^{j(\omega \cdot t - k \cdot z)} \qquad (4b)$$

$$B_2(t,z) = D_2(k) \cdot e^{j(\omega \cdot t - k \cdot z)} \qquad (4c)$$

This gives a system of linear homogeneous equations for the emergent mode amplitudes: $C(k)$, $D_1(k)$ and $D_2(k)$
These homogeneous equations have non- zero solutions if and only if the following **dispersion equation** is verified:

$$\left(\frac{\omega^2}{v_E^2} - k^2 \right) \cdot \Delta_1(k) \cdot \Delta_2(k) = k^4 \left(K_1^2 \cdot \Delta_2(k) + K_2^2 \cdot \Delta_1(k) \right) \qquad (5)$$

By definition: $\Delta_1(k) = \frac{\left(\omega^2 - \omega_{C1}^2\right)}{v_T^2} - k^2$ (6a) $\Delta_2(k) = \frac{\left(\omega^2 - \omega_{C2}^2\right)}{v_T^2} - k^2$ (6b)

When both coupling constants are different from zero, this dispersion equation always has three real, positive and non-intersecting solutions $\omega = \omega_\mu(k)$ defined for every real k with μ = l (lower), m (middle), u (upper) (Suárez-Ántola and Suárez-Bagnasco, 2001).
If k approaches zero, the coupling of the modes disappears.
The lower branch $\omega = \omega_l(k)$ approaches the dispersion equation of the uncoupled extensional mode: $\omega = v_E \cdot k$
The middle one $\omega = \omega_m(k)$ approaches the dispersion equation of the uncoupled first shear mode and the upper branch $\omega = \omega_u(k)$ approaches the dispersion equation of the second shear mode(Suárez-Ántola and Suárez-Bagnasco, 2001).

Equations (1) and (2) can be applied in the **linear hereditary** case substituting the operators $\hat{E}$ and $\hat{G}$ by the corresponding **linear operators** $\hat{E} = E_0\left(\hat{I} - \hat{L}\right)$ and $\hat{G} = G_0\left(\hat{I} - \hat{M}\right)$.

The integral operators $\hat{L}$ and $\hat{M}$ act in the time domain according to Boltzmann's principle of superposition (Rabotnov, 1980, chapters 1 and 2).
If $L(t)$ and $M(t)$ are the kernels of the operators $\hat{L}$ and $\hat{M}$ respectively, if $h(t,z)$ is a field, then by definition:

$$\hat{L}\left[h\right] = \int_{-\infty}^{t} L\left(t-u\right).h\left(u,z\right).du \tag{7a}$$

$$\hat{M}\left[h\right] = \int_{-\infty}^{t} M\left(t-u\right).h\left(u,z\right).du \tag{7b}$$

In this case it is convenient to apply the method of Finite Fourier Transforms (Zauderer, 2011).
All the fields are represented as linear combinations of a complete set of known orthonormal eigenfunctions weighted by time unknown dependent amplitudes.
In our case the eigenfunctions $\varphi_n(z)$ verify the equation $-\frac{d^2\phi_n}{dz^2} = k_n{}^2.\phi_n$ with suitably chosen homogeneous boundary conditions at the beginning $z = 0$ and the end $z = b$ of the rod, so that the given stress or displacement at the boundary of the rod or tube can be taken into account in the dynamic equations for the amplitudes.
If $h(t,z)$ represents any field (displacements, strains, stresses, or volume force densities) we define its projections onto the eigenfunctions: $h_n(t) = \int_0^b h(t,z).\varphi_n(z).dz$

Then, the field can be thus decomposed: $h(t,z) = \sum_n h_n(t).\varphi_n(z)$ (Further developments of the linear hereditary case can be found in Suárez-Antola, 2007)

3. RESULTS FOR LOW AND HIGH FREQUENCIES

3.1- Mode amplitudes, phase and group velocities in the linear elastic case for two coupled modes

To study longitudinal vibrations, either forced or free, and low frequency wave propagation, or at the other extreme, very high frequency wave propagation, a lower and an upper branch could be enough.

So, let us begin by coupling only two modes of propagation: the extensional mode with dispersion equation $\omega = v_E .k$, and one of the shear modes with dispersion equation: $\omega^2 = {\omega_c}^2 + {v_T}^2 .k^2$

We put: $K_2 = 0$ $K_1 = K$ $B_1(t,z) = B(t,z)$ $D_1(k) = D(k)$

Then the dispersion equation reduces to (now $\Delta(k) = \frac{(\omega^2 - \omega_C^2)}{v_T^2} - k^2$):

$$\left(\frac{\omega^2}{v_E^2} - k^2\right) \cdot \Delta(k) = k^4 \cdot K^2 \tag{8}$$

The system of linear equations for the mode amplitudes $C(k)$ $D(k)$ reduces to:

$$\left(\frac{\omega^2}{v_E^2} - k^2\right) \cdot C(k) = -k^2 \left(K \cdot D(k)\right) \tag{9a}$$

$$\Delta(k) \cdot D(k) = -k^2 K \cdot C(k) \tag{9b}$$

When the coupling constant is different from zero, this dispersion equation always has two real, positive and non-intersecting solutions $\omega = \omega_\mu(k)$ defined for every real k, with $\mu = l$ (lower), u (upper).

If k **approaches zero**, the lower branch $\omega = \omega_l(k)$ approaches the dispersion equation of the uncoupled extensional mode $\omega = v_E \cdot k$.

The upper branch $\omega = \omega_u(k)$ approaches the dispersion equation of the shear mode, also when k tends to zero.

But when k **tends to infinity**, the lower branch is asymptotic to $\omega = v_S \cdot k$, and if we select K properly, the upper branch will be asymptotic to $\omega = v_L \cdot k$.

The asymptotic phase velocities v_S , v_L can be found from equation (6).

If $v_a = \lim_{k \to +\infty} \left(\omega / k \right)$, taking the limit in the dispersion equation we obtain:

$$\left(\frac{v_a^2}{v_T^2} - 1\right) \cdot \left(\frac{v_a^2}{v_E^2} - 1\right) = K^2 \tag{10}$$

If we establish that $v_a = v_L$ must be a positive root of Equation (8), then we must choose

$$K^2 = \left(\frac{v_L^2}{v_T^2} - 1\right) \cdot \left(\frac{v_L^2}{v_E^2} - 1\right) \tag{11}$$

This coupling constant is a function of Poisson's modulus only.

Taking into account the well-known relations between Young's and Poisson's moduli, from one side, and v_L, v_T and v_E from the other, it follows from Equation (11) that:

$$K=\left(\frac{\nu}{1-2\nu}\right)\cdot\sqrt{\frac{2}{1+\nu}}$$

Most solids have values of ν between 0.25 and 0.30. So K increases when ν increases and verifies: $0.632 < K < 0.930$

If $v_a = v_L$ the second positive root of Equation (5) is $v_a = v_S$ and it verifies:

$$v_S^2 = v_E^2 + v_T^2 - v_L^2 \qquad (12)$$

This is the abovementioned asymptotic phase velocity of the lower branch. As will be pointed in the discussion, it should be equal to the velocity of Rayleigh waves in the material.

If $\omega = \omega_\mu(k)$ (μ = l, u) represents the two branches of the dispersion Equation (8), and if we introduce the phase velocity: $v_{f,\mu} = \omega_\mu(k)/k$ then for each mode of propagation the following relation is obtained between $C(k)$ and $D(k)$:

$$\frac{D_\mu(k)}{C_\mu(k)} = -\frac{1}{K}\left[\left(\frac{v_{f,\mu}}{v_E}\right)^2 - 1\right] \qquad (13)$$

Note that $C_\mu(k)$ and $D_\mu(k)$ always remain bounded.

In order to take into due account Poisson´s effect (axial elongation (compression) appears together with lateral contraction (dilatation)), K must be positive.

Thus if $v_{f,\mu} > v_E$, and if $C_\mu(k)$ and $D_\mu(k)$ are taken as real functions, then they have opposite signs. If they are complex functions of k, their arguments are equal or differ in π, according to the sign of the right hand member of Equation (13).

For $\mu = l$ (the lower extensional-surface mode), when k approaches to zero (long wavelengths) the phase velocity $v_{f,l}$ approaches to v_E, so that $D_l(k)$ approaches to zero.

For $\mu = u$, the upper shear-dilatational mode $D_u(k)/C_u(k)$ approaches to $-\sqrt{\left(\frac{v_L^2}{v_E^2}\right)-1\Big/\left(\frac{v_L^2}{v_T^2}\right)-1}$ when k tends to infinity (short wavelengths).

To obtain this last result we must substitute K in Eq. (13) by its expression as a function of v_L, v_T and v_E given by Equation (11).

If we define non-dimensional variables $x = R\omega/v_L$ and $y = k\cdot R$, the dispersion equation (Equation (8)) can be rewritten as follows, with $\omega_C = \alpha_T v_T/R$:

$$\left(\frac{v_L^2}{v_E^2}x^2 - y^2\right)\cdot\left(\frac{v_L^2}{v_T^2}x^2 - \alpha_T^2\right) = K^2\cdot y^4 \qquad (14)$$

From Equation (14) we can obtain x as a function of y (that is, ω as a function of k) or y as a function of x (that is, k as a function of ω).
We are going to solve ω as a function of k, for purpose of comparison with the other approaches to the dispersion relation. Thus, we obtain two branches, $f_l(y)$ and $f_u(y)$ being

$$f_{\substack{l\\(u)}}(y) = \sqrt{\frac{1}{2}\left[\alpha y^2 + \beta \underset{(+)}{-} \sqrt{\left(\beta + \gamma y^2\right)^2 + \delta y^4}\right]} \tag{15}$$

Here: $\alpha = \left(\frac{v_E}{v_L}\right)^2 + \left(\frac{v_T}{v_L}\right)^2 \quad \beta = \alpha_T^2 \left(\frac{v_T}{v_L}\right)^2 \quad \gamma = \left(\frac{v_T}{v_L}\right)^2 - \left(\frac{v_E}{v_L}\right)^2 \quad \delta = 4K^2 \left(\frac{v_E}{v_L}\right)^2 \left(\frac{v_T}{v_L}\right)^2$

Figure 1 shows the two branches of the simplified dispersion equation, calculated from Equation (15), with α_T = 3.83 and for the case of a steel bar: v_T = 3230 m/s, v_E = 5192 m/s and v_L = 5900 m/s.

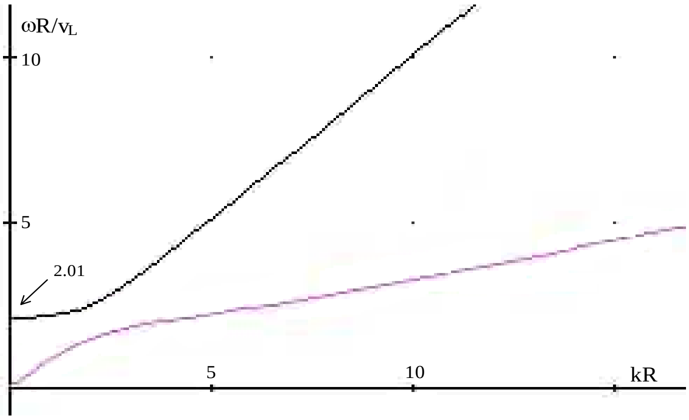

Figure 1. The dispersion relations: upper and lower branch.

Figure 2 shows the dimensionless group velocity $v_g / v_L = \left(d/dy\right) f(y)$ is a function of y for each branch of the dispersion equation, where $f(y)$ is $f_l(y)$ or $f_u(y)$ As $y = kR$ we can obtain from these curves not only how the group velocity depends of the wave-number k, but also how it depends of the bar radius R.

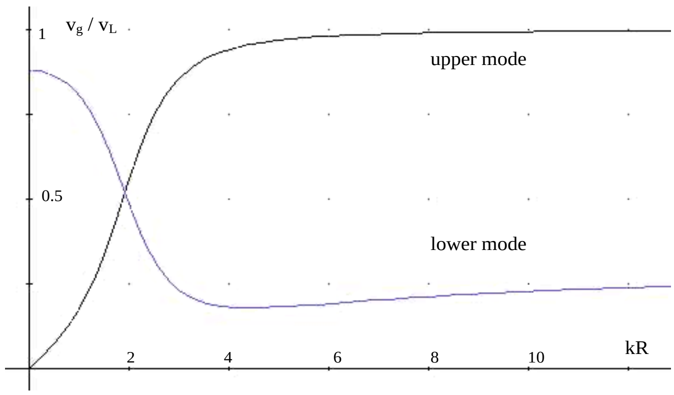

Figure 2. Upper and lower mode: group velocities as functions of the product of the radius and wave number.

The curve that gives the group velocity as a function of $k \cdot R$ for the **extensional-surface mode** present a minimum exactly as the corresponding curve found by a detailed numerical analysis of the solutions of the Pochhammer-Chree´s exact dispersion equation.
The other curve, that gives the group velocity as a function of $k.R$ for the **shear-dilatational mode**, begins from zero and grows towards a horizontal asymptote that corresponds to $v_g = v_L$.

3.2- Vibration and waves in the elastic case with two coupled modes of propagation

In principle we could consider three idealized situations:

-First, we have a wave guide that may be considered infinite in both directions. Then, for t = 0 we assume that we know the spatial pattern of mode amplitudes of the **extensional-surface** mode $A(0,z)$, and the **shear-dilatational** mode $B(t,z)$ and after that we want to follow the propagation of the elastic disturbance.
The localized initial disturbance (a wave-packet) can be expressed as a Fourier integral using the wave number k. Then, to follow the propagation of the pulse we need ω as a function of k.

-Second we have a half infinite waveguide, with an emitting transducer located at z = 0. Now we know the fields as a function of t, and we can express them as Fourier integrals using the frequency ω. To follow the propagation of the disturbance thus generated, we need k as a function of ω.

-Third situation, we have a finite rod vibrating with given boundary conditions at both ends, and we obtain a numerable infinity of possible k values.

In our case, the second and third situations are the most interesting.
Almost all the numerical calculations done for plates, rods and other types of solid waveguides give ω as a function of k, and then give both phase and group velocities as functions of k.
This allows us to calculate the frequencies of rod vibration for each possible wave number.
For **free vibrations**, the fields $A(t,z)$ and $B(t,z)$ are represented by a linear combination (series) of standing waves for the allowed wave-numbers k_n, with the frequencies of the lower and upper modes given by Eq.(12) as well defined functions of k_n.
The corresponding mode amplitudes are determined from the initial conditions and the restriction imposed by Equation (13).

On the other side, for each mode of propagation the relation between ω and k is given by a strictly monotonic function (see Fig.1).
So, instead of solving Equation (10) to find ω as a function of k it is possible to solve it to find k as a function of ω:

$$2.\left(1-K^2\right).k_\mu^{\ 2}(\omega)=k_E^{\ 2}(\omega)+k_T^{\ 2}(\omega)+\delta_\mu.\sqrt{\left(k_E^{\ 2}(\omega)+k^2{}_T(\omega)\right)^2-4.\left(1-K^2\right)k^2{}_E(\omega).k_T^{\ 2}(\omega)} \tag{16}$$

($\delta_\mu = +1$ for the lower mode $\mu = l$, and $\delta_\mu = -1$ for the upper mode $\mu = u$).

Here: $k_E^{\ 2}(\omega) = \dfrac{\omega^2}{v_E^{\ 2}}$ $k_T^{\ 2}(\omega) = \dfrac{\omega^2 - \omega_c^2}{v_T^{\ 2}}$

We can calculate $C_\mu(\omega)$ and $D_\mu(\omega)$ the same as before, now putting k as a function of the angular frequency ω .
Then we can apply these last relations to the half infinite waveguide mentioned before, adding the contributions of the lower and upper modes to construct a wave-packet.

In terms of the frequencies, the wave fields of the coupled modes are given by the following equations:

$$A(t,z)=\int_{-\infty}^{+\infty} C_l(\omega)\cdot e^{i\cdot\varphi_{l,C}(\omega)}\cdot e^{i\cdot(\omega\cdot t-z\cdot k_l(\omega))}d\omega+\int_{-\infty}^{+\infty} C_u(\omega)\cdot e^{i\cdot\varphi_{u,C}(\omega)}\cdot e^{i.(\omega.t-z.k_u(\omega))}\cdot d\omega \tag{17a}$$

$$B(t,z)=\int_{-\infty}^{+\infty} D_l(\omega)\cdot e^{i\cdot\varphi_{l,D}(\omega)}\cdot e^{i\cdot(\omega\cdot t-z\cdot k_l(\omega))}\cdot d\omega+\int_{-\infty}^{+\infty} D_u(\omega)\cdot e^{i\cdot\varphi_{u,D}(\omega)}\cdot e^{i\cdot(\omega\cdot t-z\cdot k_u(\omega))}\cdot d\omega \tag{17b}$$

In this case both $C_\mu(\omega)$ and $D_\mu(\omega)$ for $\mu = l,u$ are positive. The arguments $\phi_{\mu,C}$ and $\phi_{\mu,D}$ are either equal or differ in π by the same reasons explained before, in relation with Equation (13).
When t is big enough, the method of stationary phase (Skudrzyk, 1971) can be applied to the evaluation of the integrals (17). For given values of (t , z) the main contribution to the integral comes from the frequencies $\omega = \omega_e(t,z)$ that verify:

$$\frac{\partial k_\mu(\omega_e)}{\partial\omega} = \frac{t}{z} \qquad \mu = l,u \tag{18}$$

If (18) does not have real roots, neither for $\mu = l$ nor for $\mu = u$, both mode amplitudes A and B are negligible.

Applying the method of the stationary phase, the scalar field (17a) can be approximated by:

$$A(t,z) \approx \frac{2\cdot C_l\left(\omega_{e,l}\right)}{\left[\frac{1}{2\pi}\cdot\left|\frac{\partial^2 k_l\left(\omega_{e,l}\right)}{\partial\omega^2}\right|\cdot z\right]^{\frac{1}{2}}}.\cos\left(\omega_e\cdot t-k_l\left(\omega_{e,l}\right)\cdot z+\varphi_l\left(\omega_{e,l}\right)+s_l\cdot\frac{\pi}{4}\right)$$
$$+\frac{2\cdot C_u\left(\omega_{e,u}\right)}{\left[\frac{1}{2\pi}\cdot\left|\frac{\partial^2 k_u\left(\omega_{e,u}\right)}{\partial\omega^2}\right|\cdot z\right]^{\frac{1}{2}}}\cdot\cos\left(\omega_{e,u}\cdot t-k_u\left(\omega_{e,u}\right)\cdot z+\varphi_u\left(\omega_{e,u}\right)+s_u\cdot\frac{\pi}{4}\right) \quad (19)$$

Here s_μ is the sign of $\frac{\partial^2 k_\mu(\omega_e)}{\partial\omega^2}\neq 0$.

For the other field, (17b), we have the same expression but with D_μ instead of C_μ.

Formula (19) is a local approximation by a harmonic wave of frequency ω_e and wave number $k(\omega_e)$.

But when z/t varies, ω_e changes as well (according to formula (18)), so that we obtain a wave modulated in amplitude, frequency and phase.

When $\frac{\partial^2 k_\mu(\omega_e)}{\partial\omega^2}\approx 0$, the asymptotic method of the stationary phase fails, and another approach must be applied, like the Airy phase approximation method (Skudrzyk, 1971; Suárez-Antola, 1998).

The upper branch for high frequencies ($k.R$ much bigger than 1) may be approximated by: $\omega^2=\omega_{MS}{}^2+v_L{}^2.k^2$

Here $\omega_{MS}=\frac{\gamma.v_L}{R}$ (γ is a dimensionless parameter) is the Mc- Skimin frequency (Suárez-Antola, 1998).

For this asymptotic dispersion relation, the local frequency of the stationary phase approximation (Equation (18)) of a propagating wave packet is:

$$\omega_e(t,z)=\frac{\omega_{MS}}{\sqrt{1-\left(\frac{z}{v_L.t}\right)^2}}$$

(We have a single real root if z less than $v_L.t$, and no root and no wave at all if it is greater).

Let $\omega_{e,u}$ be the biggest frequency whose amplitude is over the detection threshold of the measurement system.

Then from the equation for $\omega_e(t,z)$ it follows that for a certain position z of the receiving transducer, the first signal detected would be registered in an instant t such that:

$$\frac{(z/t)}{v_L} = \sqrt{1 - \frac{\omega_{MS}^2}{\omega_{e,u}^2}}$$

This would give us the **apparent dilatational pulse velocity**, measured with a propagating wave-packet. Usually $\omega_{e,u}$ is greater than the carrier frequency ω_0 of the pulse.

4. CONCLUSIONS

4.1- The analytical model of wave propagation and vibrations of linear elastic solids described in the present work may be considered as an improvement and a significant generalization of the old theory of Giebe and Blechschmidt (a summary of this theory can be found in: Kolsky, 1963; Suárez-Antola and Suárez-Bagnasco, 2001).

The present model **retains certain advantages of the abovementioned old theory**: description of the longitudinal oscillations using a few modes, very good behaviour at relatively low frequencies, and applicability to bars and tubes of any cross section (after a suitable selection of the cut-off frequencies of the basic shear modes).

At the same time, the present ROM **avoids the main pitfalls** of the theory due to Giebe and Blechschmidt: death interval of frequency for rods with circular cross-section, horizontal asymptote in the ω-k plane for the lower mode, and shear wave velocity as asymptotic phase velocity for the upper mode.

For a detailed test of the results obtained, new experiments and digital simulations should be done.

4.2-As the proposed ROM can be applied to linear hereditary materials, it allows us to take into account friction processes and could be used as an intermediate step towards a generalization of the concept of region of influence of defects proposed by the author (Suárez-Antola, 2005).

4.3- The model must be completed in order to obtain a simple specification of the distribution of elastic energy between the different modes of propagation in the high frequency case.
For low frequency vibrations of rods and tubes this can be considered as already done through the volume forces and the boundary conditions that must be applied to Equations (1) and (2).
Then, with this addition and with the use of asymptotic methods, it should be possible to predict the propagation of longitudinal pulses in different solid waveguides and for different inputs.

4.4- The full calculation for the amplitudes, phase, and group velocities with three modes of propagation remains to be done. In the case of the model with three modes, we need another relation between the phenomenological constants K_1 and K_2.
The results should be tested by suitable designed experiments and digital simulations.

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